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THE RING AS A TORSION-FREE COVER

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ABSTRACT

Let R be an integral domain and I a non-zero ideal of R. The canonical map $R \to R/I$ is called a *torsion-free cover* of R/I if every R-homomorphism from a torsion-free R-module into R/I can be factored through R. The main result of this paper is that $R \to R/I$ is a torsion-free cover if and only if R is complete in the R-topology and I is an ideal of injective dimension 1. In this case I is contained in the Jacobson radical of R. And if Λ is the endomorphism ring of I, then Λ is a quasi-local domain. If I is a flat R-module, then $Q \to Q/\Lambda$ is a torsion-free cover, where Q is the quotient field of R. And then Q/Λ is an indecomposable injective R (and Λ) module. Special results are obtained if R is a Noetherian domain or a Prüfer domain.

§1. Introduction

Throughout this paper R will be an integral domain, not a field, with quotient field Q. An R-module A is said to be a *torsion-free cover* of an R-module B if (1) A is torsion-free; (2) There exists an R-homomorphism $\phi: A \to B$; (3) Ker ϕ contains no non-zero pure submodule of A; (4) If X is a torsion-free R-module and $f: X \to B$ is an R-homomorphism, then there exists an R-homomorphism $\lambda: X \to A$ such that $\phi\lambda = f$; that is, the diagram

$$\begin{tikzcd} & X \arrow[dl, "\lambda"'] \arrow[d, "f"] \\ A \arrow[r, "\phi"] & B \end{tikzcd}$$

exists and is commutative. We remark that the definition of purity used is that D is said to be *pure* in A if A/D is torsion-free. It is obvious that ϕ is a surjection.

In [3] Enochs defined and proved the existence and uniqueness of a torsion-free cover for any R-module B. In a later paper [1] Banaschewski gave a concrete construction of the torsion-free cover and an improved proof of its

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uniqueness. Nevertheless, the difficulty of determining what the torsion-free cover looks like in concrete cases remains a distinct problem. However, in [4], Enochs proved the remarkable theorem that $R \to R/M$ is a torsion-free cover for some maximal ideal M of R if and only if R is a maximal valuation ring. Cheatham provided still further insight in [2] by showing that if T is a proper, non-zero submodule of Q, then $Q \to Q/T$ is a torsion-free cover if and only if Q/T is injective and T is complete in the R-topology.

(We recall that if C is an R-module, then the R-topology on C is defined by taking all of the submodules of the form JC, where J is a non-zero ideal of R, as a base of neighborhoods of 0 in C. The completion of C is then obtained in the usual way; and C is said to be *complete* in the R-topology if it is isomorphic to its completion.)

The aim of this paper is to extend our knowledge of torsion-free covers, and to place the results of Enochs and Cheatham in greater perspective. The main theorem we shall prove is that if I is a proper, non-zero ideal of R, then $R \to R/I$ is a torsion-free cover if and only if R is complete in the R-topology, and the injective dimension of I is one.

This latter pair of conditions is the same as the condition that $\mathrm{Ext}^1_R(X, I) = 0$ for all torsion-free R-modules X. And this condition is clearly the same as the condition that whenever I is a pure submodule of a module it is a direct summand. Obviously then, an ideal I with any of these properties is special, and non-complete rings do not have any ideals of this type.

It is not hard to show that if $\mathrm{Ext}^1_R(X, I) = 0$ for all torsion-free X, then $R \to R/I$ is a torsion-free cover. This proves half of the main theorem. What is highly non-obvious, and indeed difficult to prove, is the converse. The proof of the converse occupies the main portion of this paper. Another way of stating the converse is that if the canonical map $\mathrm{Ext}^1_R(X, I) \to \mathrm{Ext}^1_R(X, R)$ is a monomorphism for all torsion-free X, then $\mathrm{Ext}^1_R(X, I) = 0$ for all torsion-free X. A careful examination of the insights obtained in the proof enables us to obtain further interesting results for special cases.

An important result obtained along the way is that if $R \to R/I$ is a torsion-free cover of R/I, then I is contained in the Jacobson radical of R. Also, using this theorem we can show that if P is a non-zero prime ideal of R, then $R \to R/P$ is a torsion-free cover if and only if $P = R_pP$ and R_p is a maximal valuation ring. As a further corollary we can show that if R is a Noetherian domain, then $R \to R/I$ is a torsion-free cover if and only if R is a complete, Noetherian, local domain of Krull dimension one; and in this case I is isomorphic to the canonical ideal of R. We also provide some new examples of torsion-free covers.

§2. Preliminaries

We now record without proof the following elementary proposition about R-submodules of Q.

PROPOSITION 0. *Let $A \neq 0$ be an R-submodule of Q. Then*

(1) *A has no proper, non-zero, pure submodules.*

(2) *Any R-module endomorphism of A is given by multiplication by a unique element of Q. Therefore, $\mathrm{Hom}_R(Q, Q) \cong Q$.*

(3) *If $A \neq Q$, then $\mathrm{Hom}_R(Q, A) = 0$ and $\mathrm{Hom}_R(Q/A, Q) = 0$.*

The following homological description of completeness ([8, theorem 9]) will be used repeatedly.

THEOREM A. *Let A be a torsion-free R-module. Then A is complete in the R-topology if and only if $\mathrm{Hom}_R(Q, A) = 0$ and $\mathrm{Ext}^1_R(Q, A) = 0$ (i.e., if and only if A is a cotorsion R-module). Thus if A is complete, then $\mathrm{Ext}^1_R(Q \otimes_R X, A) = 0$ for every R-module X.*

For the sake of easy reference we shall state without proof the following fundamental theorem of Banaschewski ([1, proposition 1, proposition 2]).

THEOREM (Banascheski). *Let B be an R-module and E an injective envelope of B. Let $T(B) = \{f \in \mathrm{Hom}_R(Q, E) \mid f(1) \in B\}$. Define ϕ: $T(B) \to B$ by $\phi(f) = f(1)$. Then ϕ: $T(B) \to B$ is a torsion-free cover of B, and every other torsion-free cover of B is isomorphic to $T(B)$.*

We shall however give a complete proof of the following theorem due to Cheatham ([2]).

THEOREM (Cheatham). *Let C be an R-module that is not torsion-free. Then the following statements are equivalent:*

(1) *C is an injective R-module, and $\mathrm{Hom}_R(Q, C) \cong Q$.*

(2) *$C \cong Q/T$ for some proper, non-zero R-submodule T of Q; T is complete in the R-topology; and $\mathrm{inj.dim}_R\, T = 1$.*

(3) *$C \cong Q/T$ for some proper, non-zero R-submodule T of Q; and $\mathrm{Ext}^1_R(X, T) = 0$ for all torsion-free R-modules X.*

(4) *$C \cong Q/T$ for some proper, non-zero R-submodule T of Q; and $Q \to Q/T$ is a torsion-free cover of Q/T.*

In this case C is an indecomposable, injective R-module.

PROOF. (1) $\Rightarrow$ (2). We have an exact sequence:

$$0 \to \mathrm{Hom}_R(Q/R, C) \to \mathrm{Hom}_R(Q, C) \to \mathrm{Hom}_R(R, C) \to 0.$$

Because $\mathrm{Hom}_R(Q, C) \cong Q$ and $\mathrm{Hom}_R(R, C) \cong C$, we see that $\mathrm{Hom}_R(Q/R, C)$ is isomorphic to a proper, non-zero R-submodule T of Q; that $C \cong Q/T$; and that $\mathrm{inj.dim}_R T = 1$.

Since $\mathrm{Hom}_R(Q, T) = 0$, by Proposition 0, we have an exact sequence:

$$0 \to \mathrm{Hom}_R(Q, Q) \to \mathrm{Hom}_R(Q, Q/T) \to \mathrm{Ext}^1_R(Q, T) \to 0.$$

But $\mathrm{Hom}_R(Q, Q) \cong Q$ by Proposition 0, and $\mathrm{Hom}_R(Q, Q/T) \cong Q$ by assumption, and thus $\mathrm{Ext}^1_R(Q, T) = 0$. It follows from Theorem A that T is complete in the R-topology.

(2) $\Rightarrow$ (3). Let X be a torsion-free R-module; then we have an exact sequence:

$$\mathrm{Ext}^1_R(Q \otimes_R X, T) \to \mathrm{Ext}^1_R(X, T) \to \mathrm{Ext}^2_R(Q/R \otimes_R X, T).$$

Now the first term of this sequence is 0 by Theorem A; and the last term is 0 because $\mathrm{inj.dim}_R T = 1$. Thus we see that $\mathrm{Ext}^1_R(X, T) = 0$.

(3) $\Rightarrow$ (4). Let X be a torsion-free R-module; then we have an exact sequence:

$$0 \to \mathrm{Hom}_R(X, T) \to \mathrm{Hom}_R(X, Q) \to \mathrm{Hom}_R(X, Q/T) \to 0.$$

Since no proper, non-zero submodule of Q is pure in Q, this sequence suffices to show that the canonical map $\pi: Q \to Q/T$ is a torsion-free cover of Q/T.

(4) $\Rightarrow$ (1). It follows from the definition of torsion-free cover that

$$0 \to \mathrm{Hom}_R(Q, T) \to \mathrm{Hom}_R(Q, Q) \to \mathrm{Hom}_R(Q, Q/T) \to 0.$$

Because $\mathrm{Hom}_R(Q, T) = 0$ and $\mathrm{Hom}_R(Q, Q) \cong Q$ by Proposition 0, we see that $\mathrm{Hom}_R(Q, Q/T) \cong Q$.

To prove that Q/T is an injective R-module, we consider an R-homomorphism $f: J \to Q/T$, where J is an ideal of R. Because the canonical map $\pi: Q \to Q/T$ is a torsion-free cover, there exists an R-homomorphism $\lambda: J \to Q$ such that $\pi\lambda = f$. By Proposition 0, λ is multiplication by an element q of Q, and thus we can define $g: R \to Q/T$ by $g(r) = qr + T$. It is immediate that g is an extension of f to all of R. This shows that Q/T is an injective R-module. If Q/T were decomposable, then we could not have $\mathrm{Hom}_R(Q, Q/T) \cong Q$; and thus Q/T is an indecomposable injective R-module.

PROPOSITION 1. (1) *Suppose that T is a proper, non-zero R-submodule of Q and that $h: Q \to Q/T$ is a torsion-free cover of Q/T. Then any non-zero R-homomorphism $g: Q \to Q/T$ is a torsion-free cover of Q/T.*

(2) *Suppose that I is a non-zero ideal of R and that $\phi: R \to R/I$ is a torsion-free cover of R/I. Then any surjection $k: R \to R/I$ is a torsion-free cover of R/I.*

PROOF. (1) There exists an R-homomorphism $\gamma: Q \to Q$ such that $h\gamma = g$ because $h: Q \to Q/T$ is a torsion-free cover of Q/T. Since γ is multiplication by a non-zero element of Q, γ is an isomorphism. It then follows easily that $g: Q \to Q/T$ is a torsion-free cover of Q/T.

(2) We have $k(1) = r + I$ for some $r \in R$; and because k is a surjection, it follows that $R = Rr + I$. Hence we have $1 = tr + a$ where $t \in R$ and $a \in I$. Because $\varphi: R \to R/I$ is a torsion-free cover of R/I, there exists an R-homomorphism $\nu: R \to R$ such that $\varphi\nu = k$. Now, ν is multiplication by $s \in R$; hence if $\varphi(1) = u + I$ for $u \in R$, we have $r + I = k(1) = \varphi(\nu(1)) = \varphi(s) = su + I$. Therefore, $r = su + b$, where $b \in I$. Thus we have

$$1 = tr + a = tsu + (tb + a) = tsu + c, \qquad \text{where } c = tb + a \text{ is in } I.$$

Suppose that X is a torsion-free R-module, and that $f: X \to R/I$ is an R-homomorphism. Then there exists an R-homomorphism $\delta: X \to R$ such that $\varphi\delta = f$. If $x \in X$, then $f(x) = tsuf(x) + cf(x) = tsuf(x)$ because $cf(x) = 0$. Thus $f = tsuf = tsu\varphi\delta = tu(\varphi s)\delta = tu(k\delta) = k(tu\delta) = k\lambda$, where $\lambda = tu\delta$ is an R-homomorphism from X to R. Therefore, because R contains no proper, non-zero pure submodules, $k: R \to R/I$ is a torsion-free cover of R/I.

We now wish to slightly generalize the concept of torsion-free cover as follows:

DEFINITION. Let B be an R-module, A a torsion-free R-module, and $\varphi: A \to B$ an R-homomorphism. We shall say that the pair (A, φ) is a *torsion-free lifting* of B is given any R-homomorphism $f: X \to B$, where X is a torsion-free R-module, then there exists an R-homomorphism $\lambda: X \to A$ such that $\varphi\lambda = f$. Because there exists a free R-module mapping onto B, it is clear that φ is a surjection. Moreover, the definition is obviously equivalent to the assertion that A is torsion-free and that for any torsion-free R-module X, the following sequence is exact:

$$0 \to \operatorname{Hom}_R(X, \operatorname{Ker}\varphi) \to \operatorname{Hom}_R(X, A) \xrightarrow{\varphi_*} \operatorname{Hom}_R(X, B) \to 0.$$

Of course a torsion-free cover of B is a torsion-free lifting of B. On the other hand, if rank $A = 1$, and (A, φ) is a torsion-free lifting of B, then (A, φ) is a torsion-free cover of B because A has no proper, non-zero pure R-submodules. More generally, it follows from [1, corollary proposition 1] that if (A, φ) is a torsion-free lifting of B, and if C is maximal among pure R-submodules of A contained in $\operatorname{Ker} \varphi$, then A/C is isomorphic to a direct summand of A and $(A/C, \bar{\varphi})$ is a torsion-free cover of B, where $\bar{\varphi}$: $A/C \to B$ is the map induced by φ. However, we shall not make use of this fact in this paper.

If C is an R-submodule of A, then $A \to A/C$ will always denote the canonical map.

PROPOSITION 2. *Let A be a torsion-free R-module, and $C \subset B \subset A$ R-submodules of A.*

(1) *If $A \to A/C$ is a torsion-free lifting of A/C, then $B \to B/C$ is a torsion-free lifting of B/C.*

(2) *If $A \to A/B$ is a torsion-free lifting of A/B; and $B \to B/C$ is a torsion-free lifting of B/C, then $A \to A/C$ is a torsion-free lifting of A/C.*

PROOF. (1) Let π: $A \to A/C$ and π_1: $B \to B/C$ be the canonical maps. Let X be a torsion-free R-module and f: $X \to B/C$ an R-homomorphism. We can view f as an R-homomorphism from X to A/C such that $\operatorname{Im} f \subset B/C$. Then, by assumption, there exists an R-homomorphism λ: $X \to A$ such that $\pi\lambda = f$. Now $\pi(\operatorname{Im} \lambda) = \operatorname{Im} f \subset B/C$, and hence $\operatorname{Im} \lambda \subset B$. Thus we can view λ as an R-homomorphism from X to B; and $\pi_1\lambda = \pi\lambda = f$. Therefore, $B \to B/C$ is a torsion-free lifting of B/C.

(2) Let π_2: $A \to A/B$ and π_3: $A/C \to A/B$ be the canonical maps. Then we have $\pi_3\pi = \pi_2$. Let Y be a torsion-free R-module and g: $Y \to A/C$ an R-homomorphism. Now $\pi_3 g$ is an R-homomorphism from Y to A/B, and hence by assumption there exists an R-homomorphism γ: $Y \to A$ such that $\pi_2\gamma = \pi_3 g$. Therefore, $\pi_3\pi\gamma = \pi_2\gamma = \pi_3 g$, and so $\operatorname{Im}(g - \pi\gamma) \subset \operatorname{Ker} \pi_3 = B/C$. Thus by assumption there exists an R-homomorphism δ: $Y \to B$ such that $\pi_1\delta = g - \pi\gamma$. Since π_1 is the restriction of π to B, we have $\pi\delta = g - \pi\gamma$. Therefore $g = \pi(\delta + \gamma)$, and $\delta + \gamma$ is an R-homomorphism from Y to A. Hence $A \to A/C$ is a torsion-free lifting of A/C.

PROPOSITION 3. *Let A be a torsion-free R-module, B an R-module and φ: $A \to B$ an R-homomorphism. Let S be a multiplicatively closed subset of R and suppose that the canonical map $B \to B_S$ is a monomorphism.*

(1) *If* $\varphi: A \to B$ *is a torsion-free lifting of* B *over* R, *then* $\varphi_S: A_S \to B_S$ *is a torsion-free lifting of* B_S *over both* R *and* R_S.

(2) *If* $\varphi_S: A_S \to B_S$ *is a torsion-free lifting of* B_S *over* R_S, *and if* $\varphi_S^{-1}(B) \subset A$, *then* $\varphi: A \to B$ *is a torsion-free lifting of* B *over* R.

PROOF. (1) Let X be a torsion-free R_S-module, and $f: X \to B_S$ an R_S-homomorphism. We can assume that $B \subset B_S$, and we let $Y = f^{-1}(B)$. Define g to be the restriction of f to Y. We view $g: Y \to B$ as an R-homomorphism; and hence there exists an R-homomorphism $\lambda: Y \to A$ such that $\varphi\lambda = g$.

Now $Y_S = X$, for $Y_S \subset X_S = X$ on the one hand; and on the other hand if $x \in X$, then $f(x) = b/s$, where $b \in B$ and $s \in S$, and so $sx \in Y$, showing that $Y_s = X$. We also have $g_S = f$. For the domain of g_S is $Y_S = X$, and if $x \in X$, then $x = y/s$ where $y \in Y$ and $s \in S$, so that $g_S(x) = g(y)/s = f(y)/s = f(y/s) = f(x)$.

Now the domain of λ_S is $Y_S = X$, and $\varphi_S\lambda_S = (\varphi\lambda)_S = g_S = f$. Hence ϕ_S is a torsion-free lifting of B_S over R_S. We next show that it is a torsion-free lifting of B_S over R as well.

Hence suppose that U is a torsion-free R-module and $h: U \to B_S$ an R-homomorphism. Then $U \subset U_S$; and $h_S: U_S \to (B_S)_S = B_S$ is an R_S-homomorphism. Hence, because $\varphi_S: A_S \to B_S$ is a torsion-free lifting of B_S over R_S, there exists an R_S-homomorphism $\delta: U_S \to A_S$ such that $\varphi_S\delta = h_S$. Let η be the restriction of δ to U. Since h is the restriction of h_S to U, we have for $u \in U$ that $h(u) = h_S(u) = \varphi_S(\delta(u)) = \varphi_S(\eta(u))$. Therefore $\varphi_S\eta = h$, and thus $\varphi_S: A_S \to B_S$ is also a torsion-free lifting of B_S over R.

(2) Now suppose that $\varphi_S: A_S \to B_S$ is a torsion-free lifting of B_S over R_S and that $\varphi_S^{-1}(B) \subset A$. Let V be a torsion-free R-module and $k: V \to B$ an R-homomorphism. Then $k_S: V_S \to B_S$ is an R_S-homomorphism, and hence by assumption there exists an R_S-homomorphism $\nu: V_S \to A_S$ such that $\varphi_S\nu = k_S$. Let ε be the restriction of ν to V. Since k is the restriction of k_S to V, we have for $v \in V$ that $k(v) = k_S(v) = \varphi_S(\nu(v)) = \varphi_S(\varepsilon(v))$; and thus $\varphi_S\varepsilon = k$. Therefore, $\varphi_S\ (\operatorname{Im}\varepsilon) \subset B$, and so $\operatorname{Im}\varepsilon \subset \varphi_S^{-1}(B) \subset A$. Thus ε is an R-homomorphism from V to A; and since φ is the restriction of φ_S to A, We have $k = \varphi_S\varepsilon = \varphi\varepsilon$. Therefore, $\varphi: A \to B$ is a torsion-free lifting of B over R.

REMARKS. Suppose that B is an R_S-module, A a torsion-free R_S-module, and $\varphi: A \to B$ an R_S-homomorphism. Then it follows immediately from Proposition 3 that $\varphi: A \to B$ is a torsion-free lifting of B over R_S if and only if it is a torsion-free lifting of B over R.

§3. The Main Theorem

Definition. Let I be a non-zero ideal of R. We shall say that an R-module X is *I-faithful* if the annihilator of I in X is 0.

Theorem 1. *Let I be a non-zero ideal of R. Then the following statements are equivalent*:

(1) $R \to R/I$ *is a torsion-free cover of* R/I.

(2) $Q \to Q/I$ *is a torsion-free cover of* Q/I.

(3) $\mathrm{Ext}^1_R(X, I) = 0$ *for all torsion-free R-modules X.*

(4) $\mathrm{Ext}^1_R(X, I)$ *is I-faithful for all torsion-free R-modules X.*

(5) *R is complete in the R-topology, and* $\mathrm{inj.dim}_R\, I = 1$.

Proof. The equivalence of (2) and (3) and (5) is a consequence of Cheatham's theorem. (If I is complete in the R-topology, so is R [8, theorem 9].) (3) implies (4) trivially. Assume that (4) is true, and let X be a torsion-free R-module. Because $\mathrm{Hom}_R(X, R/I)$ is annihilated by I we have an exact sequence:

$$0 \to \mathrm{Hom}_R(X, I) \to \mathrm{Hom}_R(X, R) \to \mathrm{Hom}_R(X, R/I) \to 0.$$

Therefore $R \to R/I$ is a torsion-free lifting of R/I. Because R has no proper, non-zero pure submodules, $R \to R/I$ is a torsion-free cover of R/I.

To conclude the proof of the theorem we must prove that (1) implies (2). We shall accomplish this in a series of lemmas.

Lemma 1. *Let I be a non-zero ideal of R and assume that $R \to R/I$ is a torsion-free cover of R/I. Then R, and all ideals of R, are complete in the R-topology.*

Proof. Let H be the completion of R in the R-topology. By [7, prop. 5.10] we have $H/HI \cong R/I$. Hence there exists a surjection of H onto R/I, and so $\mathrm{Hom}_R(H, R/I) \neq 0$. Since H is a torsion-free R-module we have an exact sequence:

$$0 \to \mathrm{Hom}_R(H, I) \to \mathrm{Hom}_R(H, R) \to \mathrm{Hom}_R(H, R/I) \to 0.$$

Thus $\mathrm{Hom}_R(H, R) \neq 0$. But then by [7, prop. 5.11] we have $H \cong R$, and R is complete in the R-topology. Thus $\mathrm{Ext}^1_R(Q, R) = 0$ by Theorem A. It follows readily that $\mathrm{Ext}^1_R(Q, J) = 0$ for any ideal J of R. This implies that J is complete in the R-topology by Theorem A.

LEMMA 2. *Let I be a non-zero ideal of R. Then the following statements are equivalent.*

(1) *$Q \to Q/I$ is a torsion-free cover of Q/I.*

(2) *$Q \to Q/T$ is a torsion-free cover of Q/T for all R-submodules T of Q that are isomorphic to I.*

(3) *$R \to R/J$ is a torsion-free cover of R/J for all ideals J of R that are isomorphic to I.*

(4) *$R \to R/rI$ is a torsion-free cover of R/rI for all non-zero elements $r \in R$.*

PROOF. (1) $\Rightarrow$ (2). This is an immediate consequence of Cheatham's theorem.

(2) $\Rightarrow$ (3). This follows directly from Proposition 2 (1).

(3) $\Rightarrow$ (4). This is a trivial implication.

(4) $\Rightarrow$ (1). By Lemma 1, I is complete in the R-topology. Hence by Cheatham's theorem it is sufficient to prove that $\operatorname{inj.dim}_R I = 1$. Let L be a non-zero ideal of R and choose a non-zero element $r \in L$. Then we have an exact sequence:

$$0 \to I \xrightarrow{r} I \to I/rI \to 0.$$

By Proposition 2 (1), $I \to I/rI$ is a torsion-free cover of I/rI. Therefore, we have an exact sequence:

$$0 \to \operatorname{Hom}_R(L, I) \xrightarrow{r} \operatorname{Hom}_R(L, I) \to \operatorname{Hom}_R(L, I/rI) \to 0.$$

It follows that the following sequence is exact:

$$0 \to \operatorname{Ext}^1_R(L, I) \xrightarrow{r} \operatorname{Ext}^1_R(L, I).$$

Thus r is not a zero-divisor on $\operatorname{Ext}^1_R(L, I)$. However, since $\operatorname{Ext}^1_R(L, I) \cong \operatorname{Ext}^2_R(R/L, I)$, it follows that $\operatorname{Ext}^1_R(L, I)$ is annihilated by r. Thus $\operatorname{Ext}^1_R(L, I) = 0$. As is well known, this implies that $\operatorname{inj.dim}_R I = 1$.

LEMMA 3. *Let I be a non-zero of R such that $R \to R/I$ is a torsion-free cover of R/I. Let S be a multiplicatively closed subset of R. Then $I_S \cap R = I$ if and only if $I = I_S$.*

PROOF. Suppose that $I_S \cap R = I$. Since the kernel of the canonical map $R \to R_S/I_S$ is $I_S \cap R$, it follows from our assumption that $R/I \to R_S/I_S$ is a monomorphism. Because $(R/I)_S \cong R_S/I_S$, we can conclude from Proposition 3 that $R_S \to R_S/I_S$ is a torsion-free cover of R_S/I_S over R. Since $I_S \subset R + I_S \subset R_S$, it follows from Proposition 2(1) that $R + I_S \to (R + I_S)/I_S$ is a torsion-free lifting of

$(R+I_S)/I_S$ over R. Because $R+I_S$ has no proper, non-zero, pure submodules, it is a torsion-free cover of $(R+I_S)/I_S$. Now

$$(R+I_S)/I_S \cong R/(I_S \cap R) = R/I.$$

Hence by the uniqueness of the torsion-free cover (see Banaschewski's Theorem), we see that $R+I_S$ is isomorphic to R.

Thus there exists a non-zero element q in $R+I_S$ such that $R+I_S = Rq$. Since $q \in R_S$, we have $q = r/s$ where $r \in R$ and $s \in S$. It follows that $sI_S \subset R$. Now $sI_S = I_S$, and so $I_S \subset R$. Therefore, $I = I_S \cap R = I_S$.

Lemma 4. *Let I be a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I. Then I is contained in the Jacobson radical of R.*

Proof. Let $S = \{1-a \mid a \in I\}$; then S is a multiplicatively closed subset of R. Let $r \in I_S \cap R$; then $r = b/s$, where $b \in I$ and $s \in S$. Hence $s = 1-a$ for some $a \in I$. Thus $r = b + ar \in I$, and so we have $I_S \cap R = I$. It follows from Lemma 3 that $I = I_S$.

Let M be a maximal ideal of R, and suppose that $I \not\subset M$. Then $R = M + I$, and hence we have $1 = m + a$, where $m \in M$ and $a \in I$. Thus $m = 1 - a = s \in S$; and so $1/m = 1 + a/m = 1 + a/s$ is an element of $R + I_S = R$. Therefore $1 = (1/m)\cdot m \in M$; and this contradiction shows that I is contained in every maximal ideal of R; i.e., I is contained in the Jacobson radical of R.

Lemma 5. *Let I be a non-zero ideal of R such that $R \to R/I$ is torsion-free cover of R/I. Let J be any ideal of R such that $J \subset I$; and let A be an R-submodule of Q. Then there exists a surjection of A onto R/J if and only if $A \cong R$.*

Proof. Suppose that $f: A \to R/J$ is a surjection. Let $\pi: R \to R/I$ be the canonical map and $\bar{\pi}: R/J \to R/I$ the canonical surjection induced by π. Then $\bar{\pi}f$ is a surjection of A onto R/I; and hence there exists an R-homomorphism $\lambda: A \to R$ such that $\pi\lambda = \bar{\pi}f$. Since λ is multiplication by a non-zero element of Q, λ is a monomorphism. On the other hand, since $\bar{\pi}f$ is a surjection, $\pi(\operatorname{Im}\lambda) = R/I$. Therefore, $R = \operatorname{Im}\lambda + I$. But I is contained in the Jacobson radical of R by Lemma 4, and so $\operatorname{Im}\lambda = R$. Thus λ is an isomorphism.

Lemma 6. *Let I be a non-zero ideal of R such that $R \to R/I$ is a torsion free cover of R/I. Then Q/I is an essential extension of R/I.*

Proof. Let x be a non-zero element of Q/I. We wish to show that $Rx \cap (R/I) \neq 0$. Hence suppose that $Rx \cap (R/I) = 0$. Now $x = a/b + I$, where a and b are non-zero elements of R. Since $x \notin R/I$, we have $a \notin Rb$.

We have $\mathrm{Ann}_R x = \{r \in R \mid ra/b \in I\} = \{r \in R \mid ra \in Ib\} = (Ib: a)$. Now $(Rb: a) = \{r \in R \mid rx \in R/I\}$; and since $Rx \cap R/I = 0$, we have $(Ib: a) = (Rb: a)$. Thus

$$(Ra + Ib) \cap Rb = (Rb: a)a + Ib = (Ib: a)a + Ib = Ib.$$

Hence $(Ra + Rb)/(Ra + Ib) \cong Rb/[(Ra + Ib) \cap Rb] = Rb/Ib \cong R/I$. Hence we have a surjection $(Ra + Rb) \to R/I$. Therefore, by Lemma 5, $Ra + Rb$ is isomorphic to R. Thus there exists $c \in Ra + Rb$ such that $Ra + Rb = Rc$.

We have $a = rc$, $b = tc$, and $c = ua + bv$, where r, t, u, v are in R. Thus $c = urc + vtc$, and so $1 = ur + vt$. Now $(Rb: a) = (Rtc: rc) = (Rt: r)$; and $(Ib: a) = (Itc: rc) = (It: r)$. Hence we have $(Rt: r) = (It: r)$. Since $t \in (Rt: r)$, we have $t \in (It: r)$. It follows that $r \in I$. Therefore $ur \in I$ also. Since I is contained in the Jacobson radical of R by Lemma 4, we have that $vt = 1 - ur$ is a unit in R. Therefore, t is also a unit in R. Hence $Rb = Rtc = Rc$, and so $a \in Rc = Rb$. This contradiction shows that $Rx \cap (R/I) \neq 0$; and so Q/I is an essential extension of R/I.

Proof of Theorem 1. We are now ready to conclude the proof of Theorem 1. Hence suppose that I is a non-zero ideal of R and that $R \to R/I$ is a torsion-free cover of R/I. We wish to prove that $Q \to Q/I$ is a torsion-free cover of Q/I. Let r be a non-zero element of R. By Lemma 2, it is sufficient to prove that $R \to R/rI$ is a torsion-free cover of R/rI. By Lemma 5 it is sufficient to prove that the torsion-free cover of R/rI given by Banaschewski's Theorem has rank 1.

Let $x = 1 + I$ and $y = 1/r + I$ in Q/I. Then $ry = x$, and so $Rx \subset Ry$. Moreover, we have $Rx \cong R/I$ and $Ry \cong R/rI$. Since Q/I is an essential extension of Rx by Lemma 6, it follows that Q/I is also an essential extension of Ry. Thus if C is an injective envelope of Q/I, then C is also an injective envelope of Rx and of Ry; i.e. of R/I and of R/rI.

Let

$$T_x = \{f \in \mathrm{Hom}_R(Q, C) \mid f(1) \in Rx\}$$

and

$$T_y = \{f \in \mathrm{Hom}_R(Q, C) \mid f(1) \in Ry\}.$$

Then $T_x \subset T_y$. Define $\varepsilon_x: T_x \to Rx$ by $\varepsilon_x(f) = f(1)$; and $\varepsilon_y: T_y \to Ry$ by $\varepsilon_y(f) = f(1)$. Thus ε_x is the restriction of ε_y to T_x. If $g \in \mathrm{Ker}\,\varepsilon_y$, then $g(1) = 0$, and hence $g \in T_x$. Thus we have $\mathrm{Ker}\,\varepsilon_x \subset \mathrm{Ker}\,\varepsilon_y \subset T_x \subset T_y$. Hence $\mathrm{rank}\, T_y = \mathrm{rank}\,\mathrm{Ker}\,\varepsilon_y = \mathrm{rank}\, T_x$. By Banaschewski's theorem, T_x and T_y are torsion-free covers of R/I

and R/rI, respectively. Thus by the uniqueness of the torsion-free cover, we have $T_x \cong R$. It follows that rank $T_y = 1$. Thus by Lemma 5, $T_y \cong R$. Hence, using Proposition 1, we see that $R \to R/rI$ is a torsion-free cover of R/rI. We now have by Lemma 2 that $Q \to Q/I$ is a torsion-free cover of Q/I.

Corollary 1. *Let $K = Q/R$; then the following statements are equivalent:*

(1) *$Q \to K$ is a torsion-free cover of K.*

(2) *R is complete in the R-topology, and K is an injective R-module* (i.e. $\text{inj.dim}_R R = 1$).

(3) *$R \to R/rR$ is a torsion-free cover of R/rR for any non-zero, non-unit element r of R.*

(4) *There exists a non-zero, non-unit element r of R such that $R \to R/rR$ is a torsion-free cover of R/rR.*

(5) $\text{Ext}_R^1(X, R) = 0$ *for any torsion-free R-module X.*

(6) $\text{Ext}_R^1(X, R)$ *is torsion-free for any torsion-free R-module X.*

(7) *There exists a non-zero, non-unit element r of R that is not a zero divisor on* $\text{Ext}_R^1(X, R)$ *for any torsion-free R-module X.*

In this case R is a quasi-local domain.

Proof. The equivalence of (1), (2), and (5) is a consequence of Cheatham's theorem. If we let $I = Rr$, then $Q/I \cong K$, and thus (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) follows readily from Theorem 1. The implications (5) $\Rightarrow$ (6) $\Rightarrow$ (7) are trivial; while (7) $\Rightarrow$ (1) follows from Theorem 1, because $\text{Ext}_R^1(X, R) \cong \text{Ext}_R^1(X, Rr)$.

By Lemma 4, every non-zero, non-unit element of R is in the Jacobson radical of R. Thus R has only a single maximal ideal; i.e., R is a quasi-local domain.

Remarks. Let I be a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I. By Lemma 6, Theorem 1, and Cheatham's theorem we have that Q/I is an injective envelope of R/I, and Q/I is an indecomposable injective R-module. It follows from [5, theorem 2.4] that I is an irreducible ideal of R; i.e., I is not the intersection of two properly larger ideals. A further consequence of the fact that Q/I is an essential extension of R/I is that if A is any R-submodule of Q such that $(A \cap R) \subset I$, then we have $A \subset I$. Thus if J is a prime ideal of R contained in I, then $J = JR_J$.

§4. The prime ideal determined by I

Throught this section we shall use the following notation.

Definition. Let I be a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I. Define $\Lambda = \{q \in Q \mid qI \subset I\}$; then, of course, Λ is the endomorphism ring of I over R; i.e., $\Lambda \cong \mathrm{Hom}_R(I, I)$. Define $\mathcal{M} = \{q \in Q \mid qI \subsetneqq I\}$, and $P = \{r \in R \mid rI \subsetneqq I\}$. Then P is called the *prime ideal of R determined by I*. We shall justify this terminology in the next theorem.

THEOREM 2. *Let I be a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I.*

(1) *Λ is a quasi-local domain with maximal ideal $\mathcal{M}$.*

(2) *If Γ is any ring between R and Λ, then $\Gamma \to \Gamma/I$ and $Q \to Q/I$ are torsion-free covers of Γ/I and Q/I, respectively, over the ring Γ.*

(3) *P is a prime ideal of R containing I. We have $I = I_P$, and $R_P \subset \Lambda$.*

(4) *If M is a maximal ideal of R, then $I = I_M$ if and only if $P \subset M$. Thus if $P \subset M$, then $R_M \subset \Lambda$.*

PROOF. (1) If λ is a non-zero element of Λ, then λ induces a surjection $\bar{\lambda}: Q/I \to Q/I$ with $\mathrm{Ker}\,\bar{\lambda} = (1/\lambda)I/I$. Hence $\lambda \in \mathcal{M}$ if and only if $\mathrm{Ker}\,\bar{\lambda} \neq 0$. Suppose λ_1 and λ_2 are in $\mathcal{M}$. Then $\mathrm{Ker}\,\bar{\lambda}_1 \neq 0$ and $\mathrm{Ker}\,\bar{\lambda}_2 \neq 0$. Now Q/I is an indecomposable injective R-module by Theorem 1 and Cheatham's theorem. Thus Q/I is an essential extension of every one of its non-zero submodules. Therefore, any two non-zero R-submodules of Q/I have non-zero intersection. Thus $\mathrm{Ker}\,\bar{\lambda}_1 \cap \mathrm{Ker}\,\bar{\lambda}_2 \neq 0$. Since $(\mathrm{Ker}\,\bar{\lambda}_1 \cap \mathrm{Ker}\,\bar{\lambda}_2) \subset \mathrm{Ker}\,(\overline{\lambda_1 + \lambda_2})$, we have $\mathrm{Ker}\,(\overline{\lambda_1 + \lambda_2}) \neq 0$. Therefore, $\lambda_1 + \lambda_2 \in \mathcal{M}$. It is clear from the definition of $\mathcal{M}$ that $\mathcal{M}$ is the set of non-units of Λ. It now follows that Λ is a quasi-local domain with maximal ideal $\mathcal{M}$.

(2) Let Γ be a ring between R and Λ; then $\Gamma I = I$. Let X be a torsion-free Γ-module, and suppose that $f: X \to Q/I$ is a Γ-homomorphism. Then f is also an R-homomorphism, and since $\pi: Q \to Q/I$ is a torsion-free cover of Q/I over R by Theorem 1, there exists an R-homomorphism $\varphi: X \to Q$ such that $\pi\varphi = f$. Because Q is a torsion-free and divisible R-module, it follows immediately that φ is also a Γ-homomorphism. Of course π is a Γ-homomorphism because $\Gamma I = I$. Thus $\pi: Q \to Q/I$ is a torsion-free cover of Q/I over Γ. But then, by Proposition 2, $\Gamma \to \Gamma/I$ is also a torsion-free cover of Γ/I over Γ.

(3) It is clear that $P = \mathcal{M} \cap R$; and since $\mathcal{M}$ is a maximal ideal of Λ, P is a prime ideal of R. If $s \in R - P$, then by the definition of P we have $sI = I$, and hence $(1/s)I = I$. Therefore, $I = I_P$. It is now obvious that $I \subset P$, and that $R_P \subset \Lambda$.

(4) Let M be a maximal ideal of R. If $P \subset M$, then $(R - M) \subset (R - P)$ and hence $I \subset I_M \subset I_P = I$. Conversely, suppose that $I = I_M$. If $t \in R - M$, then $tI = tI_M = I_M = I$, and so $t \in R - P$. Thus $P \subset M$, and it is clear that $R_M \subset \Lambda$.

THEOREM 3. *Let I be a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I. Let P be the prime ideal of R determined by I.*

(1) *If $r \in R - I$, then $(I: r) \subset P$.*

(2) *If $r \in R$, then $(I: r) = I \Leftrightarrow r \notin P$.*

(3) *If $r \in R$, then $(I: r) \cong I \Leftrightarrow (I: r)r = I \Leftrightarrow I \subset Rr \Leftrightarrow R \to R/(I: r)$ is a torsion-free cover of $R/(I: r)$.*

(4) *P is the set of elements of R that are zero-divisors on R/I.*

(5) *$(I: P) \neq I \Leftrightarrow Q/I \cong E(R/P)$, the injective envelope of R/P over R.*

(6) *R is a quasi-local ring with maximal ideal $P \Leftrightarrow sI = I$ for $s \in R$ implies that s is a unit in R.*

PROOF. (1) Suppose that $r \in R$, and let $c \in (I: r)$. If $c \notin P$, then $cI = I$; and hence, since $cr \in I$, we must have $r \in I$.

(2) Suppose that $r \in R - P$; then $rI = I$. Hence, if $c \in (I: r)$, then $cr \in I = rI$; and so $c \in I$. Thus we have $(I: r) = I$. Conversely, suppose that $(I: r) = I$. Then $I \cap Rr = (I: r)r = Ir$. Therefore, $(I + Rr)/I \cong Rr/(I \cap Rr) = Rr/Ir \cong R/I$. Thus there exists a surjection of $I + Rr$ onto R/I. Hence by Lemma 5, $I + Rr \cong R$. Therefore, there exists an element $t \in I + Rr$ such that $I + Rr = Rt$. Since $Rr \subset Rt$, we have $(I: t) \subset (I: r) = I \subset (I: t)$. Thus $(I: t) = I$. Now $I \subset Rt$, and so we have $I = I \cap Rt = (I: t)t = It$. Therefore, $t \notin P$. Because $I \subset P$, and $I + Rr = Rt$, it follows that $r \notin P$.

(3) Let $r \in R$; then $I \cap Rr = (I: r)r$. Thus $(I: r)r = I$ if and only if $I \subset Rr$. Now suppose that $(I: r) \cong I$. Then it follows readily from Theorem 1 and Lemma 2 that $R \to R/(I: r)$ is a torsion-free cover of $R/(I: r)$. On the other hand, suppose that $R \to R/(I: r)$ is a torsion-free cover of $R/(I: r)$. By Proposition 2, $(I + Rr) \to (I + Rr)/I$ is a torsion-free cover of $(I + Rr)/I$. But $(I + Rr)/I \cong Rr/(I \cap Rr) = Rr/(I: r)r \cong R/(I: r)$. Hence by the uniqueness of the torsion-free cover, we have $I + Rr \cong R$. Thus there exists an element $t \in I + Rr$ such that $I + Rr = Rt$. Hence $t = a + ur$, where $a \in I$ and $u \in R$; and $r = ct$, where $c \in R$. Thus we have $r = ca + cur$, and so $(1 - cu)r = ca \in I$. Therefore $(1 - cu) \in (I: r)$; but by Lemma 4, $(I: r)$ is contained in the Jacobson radical of R. Thus cu, and hence c, are units in R. Therefore $Rr = Rct = Rt$. Since $I \subset Rt$, we have $I \subset Rr$. Therefore, $I = I \cap Rr = (I: r)r$. It follows immediately that $I \cong (I: r)$.

(4) If $r \in R - P$, then $(I: r) = I$ by (2), and hence r is not a zero divisor on R/I. On the other hand, suppose that $r \in P$. Then by (2), $(I: r) \neq I$, and so r is a zero divisor on R/I. Thus P is the set of zero divisors of R on R/I.

(5) Suppose that $(I: P) \neq I$. There then exists an element $r \in R - I$ such that

$rP \subset I$; that is, $P \subset (I: r)$. By (1), we have $(I: r) \subset P$, and so $P = (I: r)$. Let $x = r + I \in Q/I$. Then $\mathrm{Ann}_R x = (I: r) = P$, and so $Rx \cong R/P$. Since Q/I is an indecomposable injective R-module by Theorem 1 and Cheatham's theorem, Q/I is an injective envelope of every one of its non-zero R-submodules. Therefore, $Q/I = E(Rx) \cong E(R/P)$.

Conversely, suppose that $Q/I \cong E(R/P)$. Then there exists an element $x \in Q/I$ such that $\mathrm{Ann}_R x = P$. Since Q/I is an essential extension of R/I by Lemma 6, there exists an element $t \in R$ such that $y = tx \neq 0$ and $y \in R/I$. Thus $y = r + I$ for some $r \in R - I$; and so $\mathrm{Ann}_R y = (I: r)$. We have $Py = tPx = 0$, and so $P \subset (I: r)$. But $(I: r) \subset P$ by (1). Thus $(I: r) = P$, $r \in (I: P)$, and $r \notin I$. Therefore, $(I: P) \neq I$.

(6) If R is a quasi-local ring with maximal ideal P, then $s \in R$ is a unit in $R \Leftrightarrow s \in R - P \Leftrightarrow sI = I$. On the other hand suppose that $sI = I$ for $s \in R$ implies that s is a unit of R. Then every element of $R - P$ is a unit of R, and so R is quasi-local with maximal ideal P.

REMARKS. Suppose that $R \to R/I$ is a torsion-free cover of R/I and that $(I: P) \neq I$. Then by Theorem 3 (5), $Q/I = E(R/P) = E$. By Theorem 1 and Cheatham's theorem we have $\mathrm{Hom}_R(Q, E) \cong Q$. It then follows from [6, proposition 6] that every valuation ring between R_P and Q is a maximal valuation ring and that the prime ideals contained in P are linearly ordered.

COROLLARY 2. *Let I be a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I; and let P be the prime ideal of R determined by I. If I is a finitely generated ideal of R, then R is a quasi-local ring with maximal ideal P.*

PROOF. Let $s \in R$ and suppose that $sI = I$. By Theorem 3 (6) it is sufficient to show that s is a unit in R. Suppose that s is not a unit in R. Then there exists a maximal ideal M such that $s \in M$. Then in R_M we have $s \in MR_M$ and $sI_M = I_M$. Since I_M is a finitely generated ideal of R_M, we have by the Nakayama Lemma that $I_M = 0$. This contradiction shows that s is a unit in R.

§5. Noetherian domains and Prüfer domains

THEOREM 4. *Let R be a Noetherian integral domain.*

(1) *There exists a non-zero ideal I of R such that $R \to R/I$ is a torsion-free cover of R/I if and only if R is a complete, Noetherian local domain of Krull dimension* 1. *In this case $Q/I = E = E(R/P)$ where P is the maximal ideal of R (and of course P is the prime ideal determined by I). Moreover $I \cong \mathrm{Hom}_R(Q/R, E)$, the canonical ideal of R; and $\Lambda = R$.*

(2) *There exists a non-zero, non-unit element r of R such that $R \to R/Rr$ is a torsion-free cover of R/Rr if and only if R is a complete, Noetherian, local, Gorenstein ring of Krull dimension* 1.

PROOF. (1) Suppose that $R \to R/I$ is a torsion-free cover of R/I. By Corollary 2, R is a local ring with maximal ideal P, the prime ideal determined by I. By Theorem 3 (4), P is the set of zero divisors on R/I, and hence P is one of the associated prime ideals of I. Thus there exists an element $r \in R - I$ such that $Pr \subset I$. Hence by Theorem 3 (5), we have $Q/I \cong E(R/P) = E$.

Let V be a valuation ring in Q dominating R (i.e., if $m(V)$ is the maximal ideal of V, then $R \subset V$ and $m(V) \cap R = P$). Then $V/m(V)$ is a vector space over R/P, and hence there exists a surjection of V onto R/P. Let r be an element of $R - I$ such that $Pr \subset I$, and let $x = r + I$ in R/I. Then $\mathrm{Ann}_R\, x = P$, and so $Rx \cong R/P$. Thus we have a non-zero R-homomorphism from V into R/I. Therefore, there exists a non-zero R-homomorphism $\lambda: V \to R$. Since λ is multiplication by a non-zero element of Q, λ is monomorphism. Therefore, V is isomorphic to an ideal of R and hence is a finitely generated R-module. But then V is the integral closure of R, and is a Noetherian valuation ring; i.e., a discrete valuation ring. It follows immediately that R has Krull dimension 1. But then the R-topology and the P-adic topology on R are the same; and hence R is complete in the P-adic topology by Lemma 1.

Let $K = Q/R$; by [8, theorem 9] we have $I \cong \mathrm{Hom}_R(K, K \otimes_R I) \cong \mathrm{Hom}_R(K, Q/I) \cong \mathrm{Hom}_R(K, E)$, the canonical ideal of R. We also have $\Lambda \cong \mathrm{Hom}_R(I, I) \cong \mathrm{Hom}_R(I, \mathrm{Hom}_R(K, K \otimes_R I)) \cong \mathrm{Hom}_R(K \otimes_R I, K \otimes_R I) \cong \mathrm{Hom}_R(Q/I, Q/I) \cong \mathrm{Hom}_R(E, E)$. And by [5, theorem 3.7] $\mathrm{Hom}_R(E, E) \cong R$. Thus we have $\Lambda = R$.

Conversely, suppose that R is a complete, Noetherian local domain of Krull dimension 1 with maximal ideal P. Then by [6, proof of theorem 4] there exists a non-zero ideal I of R such that $Q/I \cong E(R/P)$. Since I is complete in the R-topology, $Q \to Q/I$ is a torsion-free cover of Q/I by Cheatham's theorem. Therefore, $R \to R/I$ is a torsion-free cover of R/I by Proposition 2.

(2) Suppose that there exists a non-zero, non-unit element r of R such that $R \to R/Rr$ is a torsion-free cover of R/Rr. By part (1), R is a complete, Noetherian local domain of Krull dimension 1; and $Q/Rr \cong E(R/P)$, where P is the maximal ideal of R. Therefore, $K = Q/R$ is isomorphic to $E(R/P)$, and hence $\mathrm{inj.dim}_R R = 1$, that is, R is a Gorenstein ring.

Of course, if R is a complete, Noetherian local Gorenstein domain of Krull dimension 1 then $\mathrm{inj.dim}_R R = 1$, and so $K = Q/R$ is injective. Hence

$R \to R/Rr$ is a torsion-free cover of R/Rr for all non-zero, non-unit elements r of R by Corollary 1.

LEMMA 7. *Let R be an integral domain and I a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I. Suppose that I is a flat R-module. Then $Q \to Q/\Lambda$ is a torsion-free cover of Q/Λ over both R and Λ.*

PROOF. By Theorem 1, Q/I is an injective R-module; and thus we have an exact sequence:

$$0 \to \mathrm{Hom}_R(R/I, Q/I) \to \mathrm{Hom}_R(R, Q/I) \to \mathrm{Hom}_R(I, Q/I) \to 0.$$

Because I is a flat R-module, $\mathrm{Hom}_R(I, Q/I)$ is an injective R-module. Furthermore, $\mathrm{Hom}_R(R, Q/I) \cong Q/I$. Thus $\mathrm{inj.dim}_R \mathrm{Hom}_R(R/I, Q/I) = 1$.

However, $\mathrm{Hom}_R(R/I, Q/I) \cong$ Annihilator of I in $Q/I = \{q + I \in Q/I \mid qI \subset I\} = \Lambda/I$. Thus, $\mathrm{inj.dim}_R \Lambda/I = 1$. Since $\mathrm{inj.dim}_R I = 1$, we see that $\mathrm{inj.dim}_R \Lambda = 1$; that is Q/Λ is an injective R-module. Since Λ is isomorphic to an ideal of R, Λ is complete in the R-topology by Lemma 1. Thus by Cheatham's theorem, $Q \to Q/\Lambda$ is a torsion-free cover of Q/Λ over R. If X is a torsion-free Λ-module, then an R-homomorphism from X into Q is also a Λ-homomorphism. Therefore, $Q \to Q/\Lambda$ is a torsion-free cover of Q/Λ over Λ as well.

REMARKS. We observe that if I is a flat R-module, and if Γ is any ring between R and Λ, then $\Gamma I = I$, and I is a flat Γ-module by an elementary change of rings argument.

LEMMA 8. *Let R be an integral domain and I a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I; and let P be the prime ideal of R determined by I. If R_P is a valuation ring, then $R_P = \Lambda$, and R_P is a maximal valuation ring.*

PROOF. By Theorem 2 we can assume that $R = R_P$. Let $\lambda \in \Lambda$, and suppose that $\lambda \notin R$. Since R is a valuation ring, we have that $1/\lambda$ is in P, the maximal ideal of R. Thus $\lambda I \subset I$ and $(1/\lambda)I \subsetneq I$. This contradiction shows that $R = \Lambda$.

Since R is a valuation ring, I is a flat ideal of R. Therefore Q/Λ is an injective R-module by Lemma 7 and Cheatham's theorem; that is, $K = Q/R$ is an injective R-module. Furthermore, R is complete in the R-topology by Lemma 1. Therefore, R is a maximal valuation ring by [8, theorem 51].

We can now place the following theorem of Enochs in the context of the results we have obtained.

THEOREM (Enochs). *Let R be an integral domain. Then there exists a maximal ideal M of R such that $R \to R/M$ is a torsion-free cover of R/M if and only if R is a maximal valuation ring.*

PROOF. Suppose that M is a maximal ideal of R such that $R \to R/M$ is a torsion-free cover of R/M. By Lemma 4, R is a quasi-local ring with maximal ideal M. Let J be any finitely generated non-zero ideal of R. Then $J \neq MJ$ by Nakayama's Lemma. Hence J/MJ is a non-zero finite dimensional vector space over R/M. Thus there exists a surjection of J onto R/M. Thus by Lemma 5, J is a principal ideal of R. It follows immediately that R is a valuation ring. By Lemma 8, R is a maximal valuation ring.

Conversely, if R is a maximal valuation ring with maximal ideal M, then $\mathrm{Ext}^1_R(X, M) = 0$ for any torsion-free R-module X by [8, theorem 51]. Thus $R \to R/M$ is a torsion-free cover of R/M by Theorem 1.

LEMMA 9. *Let R be an integral domain and P a non-zero prime ideal of R. Then $R \to R/P$ is a torsion-free cover of R/P if and only if $P = PR_P$ and R_P is a maximal valuation ring.*

PROOF. Suppose that $R \to R/P$ is a torsion-free cover of R/P. By Theorem 3 (4), P is the prime ideal of R determined by P. Hence by Theorem 2 (3), we have $P = PR_P$. Since the canonical map $R/P \to R_P/PR_P$ is a monomorphism, it follows from Proposition 3 that $R_P \to R_P/PR_P$ is a torsion-free cover of R_P/PR_P over R_P. Then R_P is a maximal valuation ring by Enochs Theorem.

Conversely, suppose that $P = PR_P$, and that R_P is a maximal valuation ring. By Enochs Theorem $R_P \to R_P/PR_P$ is a torsion-free cover of R_P/PR_P over R_P. Let $\pi: R \to R/P$ be the canonical map. Then $\pi_P: R_P \to (R/P)_P \cong R_P/PR_P = R_P/P$. Hence we have $\pi_P^{-1}(R/P) = R + PR_P = R$. Thus by Proposition 3 (2), it follows that $R \to R/P$ is a torsion-free cover of R/P over R.

THEOREM 5. *Let R be a Prüfer domain, and I a non-zero ideal of R such that $R \to R/I$ is a torsion-free cover of R/I; and let P be the prime ideal of R determined by I. Then $R_P = \Lambda$ is a maximal valuation ring. Moreover, $R \to R/P$ is a torsion-free cover of R/P if and only if P is contained in the Jacobson radical of R.*

PROOF. Since R is a Prüfer ring, R_P is a valuation ring. Thus $R_P = \Lambda$ is a maximal valuation ring by Lemma 8. Now if $R \to R/P$ is a torsion-free cover of R/P, then P is contained in the Jacobson radical of R by Lemma 4. Conversely, assume that P is contained in the Jacobson radical of R. Let M be a maximal ideal of R. Since R is a Prüfer ring, R_M is a valuation ring. Now $P \subset M$, and hence P_M is a prime ideal of R_M and also $R_M \subset R_P$. Therefore, because both R_P and R_M are valuation rings, the maximal ideal of R_P is contained in R_M. Hence $PR_P = P_M$. Thus as M ranges over all maximal ideals of R we have $PR_P = \cap P_M$. However, it is a general fact about localizations that $\bigcap P_M = P$. Therefore, $PR_P = P$. Since R_P is a maximal valuation ring, it follows from Lemma 9 that $R \to R/P$ is a torsion-free cover of R/P over R.

Corollary 3. *Let R be a valuation ring. Then the following statements are equivalent:*

(1) *There exists a non-zero ideal I of R such that $R \to R/I$ is a torsion-free cover of R/I.*

(2) *There exists a non-zero prime ideal P of R such that $R \to R/P$ is a torsion-free cover of R/P.*

(3) *There exists a non-zero prime ideal P of R such that R_P is a maximal valuation ring.*

Proof. (1) $\Rightarrow$ (3). Lemma 8.

(3) $\Rightarrow$ (2). Lemma 9.

(2) $\Rightarrow$ (1). Trivial.

Corollary 4. *Let R be an integral domain such that R_M is a maximal valuation ring for any maximal ideal M of R. Let P be a non-zero prime ideal of R. Then P is contained in the Jacobson radical of R if and only if $R \to R/P$ is a torsion-free cover of R/P.*

Proof. If $R \to R/P$ is a torsion-free cover of R/P, then P is contained in the Jacobson radical of R by Lemma 4. Conversely suppose that P is contained in the Jacobson radical of R. As in the proof of Theorem 5, we have $P = PR_P$. If M is any maximal ideal of R, then $P \subset M$, and hence $R_P = (R_M)_P$. Since R_M is a maximal valuation ring by assumption, $(R_M)_P$ is a maximal valuation ring by [8, theorem 93]. Therefore $R \to R/P$ is a torsion-free cover of R/P by Lemma 9.

Remarks. We observe that in [8, p. 154] an example is given of a domain R that is not a valuation ring, but such that R_M is a maximal valuation ring for any maximal ideal M of R, and such that there exists a non-zero prime ideal in the Jacobson radical of R. Of course, there also exist examples of complete Noetherian local domains of Krull dimension 1 that are not discrete valuation rings. By Corollary 4 and Theorem 4, both of these examples have non-zero ideals I such that $R \to R/I$ is a torsion-free cover of R/I. Thus this condition, while indeed quite restrictive, nonetheless has many interesting examples that are not maximal valuation rings.

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